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On the Quillen determinant

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Abstract

We explain the bundle structures of the *Determinant line bundle* and the *Quillen determinant line bundle* considered on the connected component of the space of Fredholm operators including the identity operator in an intrinsic way. Then we show that these two are isomorphic and that they are non-trivial line bundles and trivial on some subspaces. Also we remark a relation of the *Quillen determinant line bundle* and the *Maslov line bundle*.

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1. Introduction

The Fredholm determinant is defined for the class of the operators of the form "Id + trace class operator" on a Hilbert space H as the extension of the finite dimensional cases with respect to the trace norm:

$$\det_F(\mathrm{Id} + K) = \prod(1 + \lambda_i), \tag{1.1}$$

where λ_i are eigenvalues of the trace class operator K (see [5] for analytic properties of the Fredholm determinant). This quantity gives us a $\mathbb{C}^*$-valued holomorphic one cocycle on the space of Fredholm operators on H whose Fredholm indexes are zero. In fact, let $\mathcal{F} = \mathcal{F}(H)$ be the space of Fredholm operators defined on a Hilbert space H and we denote by $\mathcal{F}_0 = \mathcal{F}_0(H)$ the connected component of $\mathcal{F}(H)$ consisting of the operators with the

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index zero. Let $\mathcal{I}_1$ be the space of trace class operators on H. For each trace class operator $A \in \mathcal{I}_1$, we denote by $\mathcal{U}_A$ an open subset of $\mathcal{F}_0$ consisting of such operators T that $T + A$ is an isomorphism of H. Then $\mathcal{F}_0$ is covered by open subsets $\{\mathcal{U}_A\}_{A\in\mathcal{I}_1}$. Let A and B be two trace class operators, and let $T \in \mathcal{U}_A \cap \mathcal{U}_B \subset \mathcal{F}_0$, then the functions $\{g_{A,B}(T)\}_{A,B\in\mathcal{I}_1}$:

$$g_{A,B}(T) = \det_F\{\mathrm{Id} + (A - B)(T + B)^{-1}\} = \det_F\{(T + A)(T + B)^{-1}\}$$

are holomorphic on $\mathcal{U}_A \cap \mathcal{U}_B$, and they satisfy the cocycle condition:

$$g_{A,C}(T) = g_{A,B}(T)g_{B,C}(T) \tag{1.2}$$

for $T \in \mathcal{U}_A \cap \mathcal{U}_B \cap \mathcal{U}_C$. We denote by $\mathcal{L}_F$ the complex line bundle defined by these transition functions $\{g_{A,B}\}_{A,B\in\mathcal{I}_1}$ and call it as the "*Determinant line bundle*".

The disjoint unions of finite dimensional vector spaces:

$$\coprod_{T\in\mathcal{F}_0} \mathrm{Ker}(T) \quad \text{and} \quad \coprod_{T\in\mathcal{F}_0} \mathrm{Coker}(T)$$

do not have vector bundle structures. When we consider them on a compact subset X in $\mathcal{F}_0$, it can be seen that the formal difference of these two is an element of the K-group $K(X)$ by *approximating* each of these two with suitable vector bundles which are constructed by a standard method.

On the other hand, the disjoint union of the lines:

$$\coprod_{T\in\mathcal{F}_0} \overset{\dim \mathrm{Ker}(T)}{\wedge} \mathrm{Ker}(T)^* \otimes \overset{\dim \mathrm{Coker}(T)}{\wedge} \mathrm{Coker}(T)$$

has a complex line bundle structure on the whole space $\mathcal{F}_0$ and is called as the *Quillen determinant line bundle*. This fact is stated in the paper [11] and it is treated in various contexts ([2,9,10,13] and others).

In this note, we give a rigorous proof of this fact by giving an intrinsic correspondence between the Determinant line bundle and the Quillen determinant (Section 2), and prove that it is non-trivial on $\mathcal{F}_0$ (Section 3). In Section 4 we show it is trivial on each compact subset of the subspace $\hat{\mathcal{F}}_*$ (=the non-trivial connected component in the space of selfadjoint Fredholm operators). Of course it is trivial on the each subspace of essentially positive and essentially negative Fredholm operators (=$\hat{\mathcal{F}}_\pm$). Finally in Section 5 we prove that the induced bundle of the Quillen determinant line bundle on the space of *Fredholm Lagrangian Grassmannian* by a naturally defined map is trivial and remark a relation with the *Maslov line bundle*.

2. Fredholm determinant and the Quillen determinant

Let T be a Fredholm operator on a (complex)Hilbert space H. We denote by $\mathcal{A}_T$ a subset of the space of trace class operators $\mathcal{I}_1$ such that

$$\mathcal{A}_T = \{A \in \mathcal{I}_1 | T + A \text{ is invertible}\}$$

and let $\mathcal{D}_T$ be a space of complex valued functions on $\mathcal{A}_T$ satisfying the following condition:

$$\mathcal{D}_T = \{f : \mathcal{A}_T \to \mathbb{C} | f(B) = \det_F\{(T + A)(T + B)^{-1}\}f(A)\}.$$

Of course this is an one-dimensional vector space, and the union:

$$\coprod_{T\in\mathcal{F}_0(H)} \mathcal{D}_T$$

becomes a holomorphic complex line bundle with local trivializations:

$$j_A : \coprod_{T\in\mathcal{U}_A} \mathcal{D}_T \tilde{\to} \mathcal{U}_A \times \mathbb{C}, \qquad j_A : \mathcal{D}_T \ni f \mapsto (T, f(A)) \in \mathcal{U}_A \times \mathbb{C}, \tag{2.1}$$

where $A \in \mathcal{I}_1$ and $\mathcal{U}_A = \{T \in \mathcal{F}_0(H) | T + A \text{ is invertible}\}$. By the definition of the function space $\mathcal{D}_T$, the transition function on $\mathcal{U}_A \bigcap \mathcal{U}_B$ is given by

$$\det{}_F\{(T + A)(T + B)^{-1}\},$$

so that the space

$$\coprod_{T\in\mathcal{F}_0(H)} \mathcal{D}_T$$

is a realization of the *Determinant line bundle*. We denote it by $\mathcal{L}_F$.

For a fixed $T \in \mathcal{F}_0$, we denote by $\pi_T : H \to H$ the orthogonal projection operator onto the $\mathrm{Ker}(T)$ and by ρ_T the natural projection $\rho_T : H \to \mathrm{Coker}(T)$.

Let L be a linear map $L : \mathrm{Ker}(T) \to H$ satisfying the condition:

$$\text{The composition } \rho_T \circ L : \mathrm{Ker}(T) \to \mathrm{Coker}(T) \text{ is an isomorphism.} \tag{2.2}$$

Then under this condition for the operator L we know that the operator $T + L \circ \pi_T$ is an isomorphism on H.

Let $\{e_i\}_{i=1}^d$ be a basis of $\mathrm{Ker}(T)$ and $\{e_i^*\}_{i=1}^d$ the dual basis ($d = \dim \mathrm{Ker}\, T$). Then we define a map

$$\phi_T : \mathcal{D}_T \to \overset{\dim \mathrm{Ker}(T)}{\wedge} \mathrm{Ker}(T)^* \otimes \overset{\dim \mathrm{Coker}(T)}{\wedge} \mathrm{Coker}(T) \tag{2.3}$$

by

$$\begin{aligned} \phi_T(f) &= f(A) \cdot \det{}_F\{(T + A)(T + L \circ \pi_T)^{-1}\} \\ &\quad \times e_1^* \wedge \cdots \wedge e_d^* \otimes \rho_T(L(e_1)) \wedge \cdots \wedge \rho_T(L(e_d)), \end{aligned} \tag{2.4}$$

where we fixed an $A \in \mathcal{A}_T$. By the relation

$$\begin{aligned} &f(A) \det{}_F\{(T + A)(T + L \circ \pi_T)^{-1}\} \det{}_F\{((T + B)(T + L \circ \pi_T)^{-1})^{-1}\} \\ &\quad = f(A) \det{}_F\{(T + A)(T + B)^{-1}\} = f(B), \end{aligned}$$

it will be clear of the independence of the definition of this map from the choice of $A \in \mathcal{A}_T$ and the map ϕ_T is an isomorphism. Moreover we have the following proposition.

Proposition 2.1. *The definition of the map ϕ_T depends neither on the choice of the map L satisfying the condition (2.2) above, nor on the choice of the basis $\{e_i\}$ of* $\mathrm{Ker}(T)$.

Proof. Again it would be clear of the independence from the choice of a basis of $\mathrm{Ker}(T)$. So we only prove the independence from the choice of the operator L.

Let L' be another such operator $L' : \mathrm{Ker}(T) \to H$ that $\rho_T \circ L' : \mathrm{Ker}(T) \to \mathrm{Coker}(T)$ is isomorphic, then we have

$$\rho_T(L'(e_j)) = \sum_i a_{ij}\rho_T(L(e_i))$$

and

$$T + L \circ \pi_T = T + L' \circ \pi_T \text{ on } \mathrm{Ker}(T)^{\perp}.$$

Hence

$$(T + L \circ \pi_T)^{-1} \circ (T + L' \circ \pi_T) - \mathrm{Id}$$

is a finite rank operator, and moreover we have

$$\det{}_F\{(T + L \circ \pi_T)^{-1} \circ (T + L' \circ \pi_T)\} = \det(a_{ij}).$$

This relation gives us

$$\begin{aligned} f(A) \cdot \det{}_F\{(T + A)(T + L \circ \pi_T)^{-1}\}e_1^* \wedge \cdots \wedge e_d^* \otimes \rho_T(L(e_1)) \wedge \cdots \wedge \rho_T(L(e_d)) \\ = f(A) \cdot \det{}_F\{(T + A)(T + L' \circ \pi_T)^{-1}\}e_1^* \wedge \cdots \wedge e_d^* \otimes \rho_T(L'(e_1)) \wedge \cdots \\ \wedge \rho_T(L'(e_d)), \qquad (2.5) \end{aligned}$$

which proves the independence of the definition of the map ϕ_T from the choice of the linear map L. □

By this proposition we can introduce (the topology and) the local trivialization of the space:

$$\coprod_{T \in \mathcal{U}_A} \overset{\dim \mathrm{Ker}(T)}{\wedge} \mathrm{Ker}(T)^* \otimes \overset{\dim \mathrm{Coker}(T)}{\wedge} \mathrm{Coker}(T)$$

through the local trivialization (2.1) and the map ϕ_T:

$$\left(\coprod_{T \in \mathcal{U}_A} \phi_T\right) \circ j_A^{-1} : \mathcal{U}_A \times \mathbb{C} \xrightarrow{\sim} \coprod_{T \in \mathcal{U}_A} \overset{\dim \mathrm{Ker}(T)}{\wedge} \mathrm{Ker}(T)^* \otimes \overset{\dim \mathrm{Coker}(T)}{\wedge} \mathrm{Coker}(T). \qquad (2.6)$$

Then

$$\coprod_{T \in \mathcal{F}_0} \overset{\dim \mathrm{Ker}(T)}{\wedge} \mathrm{Ker}(T)^* \otimes \overset{\dim \mathrm{Coker}(T)}{\wedge} \mathrm{Coker}(T)$$

becomes a complex line bundle which is isomorphic to the Determinant line bundle $\mathcal{L}_F$. This is the "*Quillen determinant line bundle*" and we denote it by $\mathcal{L}_Q$.

3. Non-triviality of the Quillen determinant

Theorem 3.1. *The bundle $\mathcal{L}_Q$ is not trivial on the whole space $\mathcal{F}_0$.*

Proof. For a compact Hausdorff space X we know by the famous theorem [1] that the reduced K-group $\tilde{K}(X)$ is isomorphic to the space of homotopy classes $[X, \mathcal{F}_0]$ of continuous maps $f : X \to \mathcal{F}_0$ and the correspondence is given by constructing two vector bundles E and F on X which satisfy the following exact sequence at each point $x \in X$:

$$0 \to \mathrm{Ker}(f(x)) \to E_x \to F_x \to \mathrm{Coker}(f(x)) \to 0. \tag{3.1}$$

The homotopy class of the map f corresponds to the element $[E] - [F] \in K(X)$.

Hence we have $f^*(\mathcal{L}_Q) \cong \wedge^{\dim E} E^* \otimes \wedge^{\dim F} F$, and so for any line bundle ℓ on a compact space X the element $[\ell] - [\varepsilon^1] \in \tilde{K}(X)$ (ε^1: one-dimensional trivial line bundle) corresponds to a continuous map $g : X \to \mathcal{F}_0$, we have $\ell^* \cong g^*(\mathcal{L}_Q)$. Hence we know by taking a suitable compact space X with $H^2(X, \mathbb{Z}) \neq \{0\}$ that $\mathcal{L}_Q$ cannot be trivial on the whole space $\mathcal{F}_0(H)$. □

4. A triviality of the Quillen determinant

Although we have proved that the Quillen determinant line bundle is not trivial on the whole space $\mathcal{F}_0$, it might be trivial on a subspace in $\mathcal{F}_0(H)$. For example, it is trivial on the space of essentially positive (negative) Fredholm operators ($=\hat{\mathcal{F}}_\pm$).

Now let $\hat{\mathcal{F}}_*$ be the non-trivial connected component of the selfadjoint Fredholm operators. Then we have the following theorem.

Theorem 4.1. *On each compact subset in the space $\hat{\mathcal{F}}_*$ the Quillen determinant $\mathcal{L}_Q$ is trivial.*

Proof. Let X be a compact Hausdorff space and let f be a continuous map, $f : X \to \hat{\mathcal{F}}_*$. It is enough to show that $f^*(\mathcal{L}_Q)$ is trivial. Let $\Omega\mathcal{F}_0$ be the path space consisting of paths connecting Id and $-$Id. Let $\alpha : \hat{\mathcal{F}}_* \to \Omega\mathcal{F}_0$ be a continuous map given by

$$\alpha(A)(T) = \cos(\pi t) + \sin(\pi t) \cdot A \in \Omega\mathcal{F}_0, \quad t \in [0, 1]. \tag{4.1}$$

This is a homotopy equivalence (see [3]).

Now let $S(X)$ be the suspension of X, then we have a continuous map $h_f : S(X) \to \mathcal{F}_0$ defined by

$$h_f(t, x) = \alpha(f(x))(T).$$

Let $\mathcal{C} : X \to S(X)$ be a map defined as $\mathcal{C}(x) = (1/2, x) \in S(X)$, then by the definition of the suspension we have

$$h_f \circ \mathcal{C} = \mathbf{i} \circ f,$$

where $\mathbf{i}$ is the inclusion map $\mathbf{i} : \hat{\mathcal{F}}_* \hookrightarrow \mathcal{F}_0$. Since $\tilde{K}(S(X)) = \text{ind-lim}_{n\to\infty}[X, GL(n, \mathbb{C})]$, we know that the induced map $\mathcal{C}^* : \tilde{K}(S(X)) \to \tilde{K}(X)$ is trivial. Hence the induced line bundle $(h_f \circ \mathcal{C})^*(\mathcal{L}_Q) = f^*(\mathcal{L}_Q)$ must be trivial. □

5. Quillen determinant on the Fredholm Lagrangian Grassmannian

In this section, we show that the Quillen determinant is trivial, when it is pull-backed on the *Fredholm Lagrangian Grassmannian* through an embedding.

First we describe the Fredholm Lagrangian Grassmannian. So, let H here be a real symplectic Hilbert space. The symplectic form is non-degenerate in such a sense that $\omega : H \times H \to \mathbb{R}$ defines the continuous isomorphism $\omega^{\#}$:

$$\omega^{\#} : H \to H^*, \qquad \omega^{\#}(x)(y) = \omega(x, y). \tag{5.1}$$

We do not change the symplectic form ω once it has been introduced on a real Hilbert space H, but rather freely we can replace the inner product with a new one whose defining norm is equivalent to that defined by the original inner product. Especially, we can assume from the beginning that the symplectic form ω is expressed in the form $\omega(x, y) = \langle J(x), y\rangle$, where J is an almost complex structure with the property that $\langle J(x), J(y)\rangle = \langle x, y\rangle$, ${}^tJ = -J$, ${}^tJ$ is the transpose operator with respect to the (Euclidean) inner product $\langle\cdot, \cdot\rangle$.

Let λ be a Lagrangian subspace:

$$\lambda = \lambda^{\circ} = \{x \in H | \omega(x, y) = 0 \text{ for any } y \in \lambda\} \tag{5.2}$$

and denote by $\mathcal{F}\Lambda_{\lambda}(H)$ the space of such Lagrangian subspaces μ that the pair (μ, λ) is a Fredholm pair (see [7] for a general theory of Fredholm pairs and [4] for particular properties of Fredholm pairs of Lagrangian subspaces), that is:

(i) $\dim(\lambda \cap \mu) < +\infty$,
(ii) $\lambda + \mu$ is a closed and finite codimensional subspace in H.

We call this space as the "*Fredholm Lagrangian Grassmannian*". The topology is naturally defined by embedding it into the space of bounded operators $\mathcal{B}(H)$ on H by the map $\mathcal{P} : \mathcal{F}\Lambda_{\lambda}(H) \to \mathcal{B}(H)$, $\mathcal{P}(\mu)$ is the orthogonal projection operator onto the space μ and $\mathcal{F}\Lambda_{\lambda}(H)$ becomes an infinite dimensional smooth manifold. It is known that the fundamental group $\pi_1(\mathcal{F}\Lambda_{\lambda}(H))$ is $\mathbb{Z}$ and the isomorphism is given by so called the *Maslov index* for each loop.

When we regard the real Hilbert space H as a complex Hilbert space by means of the almost complex structure J with the Hermitian inner product:

$$(x, y) = \langle x, y\rangle - \sqrt{-1}\langle J(x), y\rangle,$$

we denote it by H_J.

Each Lagrangian subspace λ defines a *real structure* on H_J:

$$\lambda \otimes \mathbb{C} \xrightarrow{\sim} H_J, \qquad x \otimes 1 + y \otimes \sqrt{-1} \mapsto x + J(y).$$

We denote by τ_{λ} the complex conjugation with respect to a real structure given by a Lagrangian subspace λ:

$$\tau_{\lambda}(x + J(y)) = x - J(y). \tag{5.3}$$

This is an anti-linear involution on $H = H_J$ and $2\mathcal{P}(\lambda) - \mathrm{Id} = \tau_{\lambda}$.

Let $\mu \in \mathcal{F}\Lambda_\lambda(H)$, then the operator

$$-\tau_\mu \circ \tau_\lambda$$

is a unitary operator $\in \mathcal{U}(H_J)$ with the property that

$$\mathrm{Id} - \tau_\mu \circ \tau_\lambda$$

is a Fredholm operator. We denote the correspondence $\mu \mapsto -\tau_\mu \circ \tau_\lambda$ by

$$\mathcal{S}_\lambda : \mathcal{F}\Lambda_\lambda(H) \to \mathcal{U}_F(H_J), \tag{5.4}$$

where $\mathcal{U}_F(H_J)$ is a space of unitary operators U on H_J such that $U + \mathrm{Id}$ is a Fredholm operator.

We call the map $\mathcal{S}_\lambda$ the *Souriau map* [4,8,12] which satisfies $\mathcal{S}_\lambda(\mu)^* = \mathcal{S}_\mu(\lambda)$. We know that through this map the fundamental groups of the Fredholm Lagrangian Grassmannian and the space $\mathcal{U}_F(H_J)$ are isomorphic.

Let us denote by q_λ the map:

$$q_\lambda : \mathcal{F}\Lambda_\lambda(H) \to \mathcal{F}_0(H_J), \qquad q_\lambda(\mu) = \mathrm{Id} - \tau_\mu \circ \tau_\lambda = \mathrm{Id} + \mathcal{S}_\lambda(\mu).$$

Theorem 5.1. *The pull back $q_\lambda^*(\mathcal{L}_Q)$ is trivial.*

Proof. For the proof it is enough to notice the basic facts relating with the Souriau map and Fredholm pairs of Lagrangian subspaces [4,8].

For $\mu \in \mathcal{F}\Lambda_\lambda(H)$ let $p_\lambda(\mu)$ be

$$p_\lambda(\mu) = \mathcal{P}(\mu^\perp) + \mathcal{P}(\lambda^\perp),$$

then it is a positive Fredholm operator. That is, we have a map:

$$p_\lambda : \mathcal{F}\Lambda_\lambda(H) \to \hat{\mathcal{F}}_+(H). \tag{5.5}$$

Then,

$$\mathrm{Ker}(p_\lambda(\mu)) = \lambda \cap \mu$$

and

$$\mathrm{Coker}(p_\lambda(\mu)) = H/(\lambda^\perp + \mu^\perp) \cong \lambda/(\lambda \cap (\lambda^\perp + \mu^\perp)).$$

Also we know

$$\mathrm{Ker}(q_\lambda(\mu)) = \mathrm{Ker}(\mathrm{Id} + \mathcal{S}_\lambda(\mu)) = \lambda \cap \mu + J(\lambda \cap \mu) \cong (\lambda \cap \mu) \otimes \mathbb{C}$$

and since $\mathrm{Im}(q_\lambda(\mu)) = \lambda \cap (\lambda \cap \mu)^\perp + J(\lambda \cap (\lambda \cap \mu)^\perp)$:

$$\begin{aligned}\mathrm{Coker}(q_\lambda(\mu)) &= H_J/(\lambda \cap (\lambda \cap \mu)^\perp + J(\lambda \cap (\lambda \cap \mu)^\perp)) \\ &\cong (\lambda/(\lambda \cap (\lambda^\perp + \mu^\perp))) \otimes \mathbb{C}.\end{aligned}$$

So the fiber of the induced bundle $q_\lambda^*(\mathcal{L}_Q)$ by the maps q_λ is the complexification of that by the map p_λ, hence the bundle $q_\lambda^*(\mathcal{L}_Q)$ is trivial, since the Quillen determinant is trivial on the subspace $\hat{\mathcal{F}}_+(H) \subset \hat{\mathcal{F}}_+(H \otimes \mathbb{C})$. □

Corollary 5.2. *The disjoint union*

$$\coprod_{\mu\in\mathcal{F}\Lambda_\lambda} \overset{\dim \lambda\cap\mu}{\wedge}(\lambda\cap\mu)^* \otimes \overset{\dim H/(\lambda^\perp+\mu^\perp)}{\wedge} H/(\lambda^\perp+\mu^\perp)$$

has a bundle structure as a line bundle on the Fredholm Lagrangian Grassmannian $\mathcal{F}\Lambda_\lambda(H)$ and is a trivial line bundle.

Note that we do not have a particular trivialization on the whole space $\mathcal{F}\Lambda_\lambda(H)$.

Remark 5.3. For any Fredholm pair (λ, μ) of Lagrangian subspaces:

$$\dim \lambda\cap\mu = \dim H/(\lambda^\perp+\mu^\perp) = \dim H/(\lambda+\mu).$$

Now let θ be a Lagrangian subspace which "*almost coincides*" with λ:

$$\dim \lambda/(\lambda\cap\theta) = \dim \theta(\lambda\cap\theta) < \infty.$$

This relation is an equivalence relation among Lagrangian subspaces and we denote it by $\lambda\sim\theta$. Then for such a pair (λ,θ), $\lambda\sim\theta$, the Fredholm Lagrangian Grassmannian coincides with each other:

$$\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\theta(H).$$

For a Lagrangian subspace θ, let us denote an open subset in $\mathcal{F}\Lambda_\theta(H)$:

$$\{\mu\in\mathcal{F}\Lambda_\theta(H) | \mu\cap\theta = \{0\}\}$$

by $\mathcal{F}\Lambda_\theta^{(0)}(H)$. Then this space is isomorphic to the space of (real) bounded selfadjoint operators on θ and we have an open covering:

$$\mathcal{F}\Lambda_\lambda(H) = \bigcup_{\theta\sim\lambda} \mathcal{F}\Lambda_\theta^{(0)}(H).$$

On each open subset $\mathcal{F}\Lambda_\theta^{(0)}(H)$ $(\theta\sim\lambda)$, we have a trivialization of the induced bundle $q_\lambda^*(\mathcal{L}_Q)$ given by the trivialization (2.6) on $\mathcal{U}_{A_\theta}$ with a trace class operator $A_\theta = -\mathrm{Id}+\tau_\theta\circ\tau_\lambda$ (in fact this is a finite rank operator). Also there is a trivialization on $\mathcal{F}\Lambda_\theta^{(0)}(H)$ coming from the trivialization on an open subset $\mathcal{U}_{\mathcal{P}(\theta^\perp)-\mathcal{P}(\lambda^\perp)} \cap \mathcal{F}(H) \subset \mathcal{F}(H\otimes\mathbb{C})$ through the map p_λ. Here again the operator $\mathcal{P}(\theta^\perp)-\mathcal{P}(\lambda^\perp) : H\to H$, is a finite rank operator. For such two θ and $\tilde\theta$ $(\theta\sim\tilde\theta)$, the transition function on the intersection $\mathcal{F}\Lambda_\theta^{(0)}(H)\cap\mathcal{F}\Lambda_{\tilde\theta}^{(0)}(H)$ is given by the function through the map q_λ:

$$\det{}_F\{(\tau_\theta-\tau_\mu)(\tau_{\tilde\theta}-\tau_\mu)^{-1}\} = \det{}_F\{(\mathcal{P}(\theta)-\mathcal{P}(\mu))(\mathcal{P}(\tilde\theta)-\mathcal{P}(\mu))^{-1}\} \tag{5.6}$$

and that through the map p_λ is

$$\det{}_F\{(\mathcal{P}(\theta^\perp)+\mathcal{P}(\mu^\perp))(\mathcal{P}(\tilde\theta^\perp)+\mathcal{P}(\mu^\perp))^{-1}\}. \tag{5.7}$$

Now we show these two functions coincide on $\mathcal{F}\Lambda_\theta^{(0)}(H)\cap\mathcal{F}\Lambda_{\tilde\theta}^{(0)}(H)$.

Proposition 5.4. *Let θ and $\tilde{\theta}$ "almost coincide", then for $\mu \in \mathcal{F}\Lambda^{(0)}_{\theta}(H) \cap \mathcal{F}\Lambda^{(0)}_{\tilde{\theta}}(H)$ we have*

$$\det{}_F\{(\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))^{-1}\} \tag{5.8}$$

$$= \det{}_F\{(\mathcal{P}(\theta^{\perp}) - \mathcal{P}(\mu^{\perp}))(\mathcal{P}(\tilde{\theta}^{\perp}) - \mathcal{P}(\mu^{\perp}))^{-1}\} \tag{5.9}$$

$$= \det{}_F\{(\mathcal{P}(\theta) + \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) + \mathcal{P}(\mu))^{-1}\}$$
$$= \det{}_F\{(\mathcal{P}(\theta^{\perp}) + \mathcal{P}(\mu^{\perp}))(\mathcal{P}(\tilde{\theta}^{\perp}) + \mathcal{P}(\mu^{\perp}))^{-1}\}. \tag{5.10}$$

Proof. Since $\mathcal{P}(x) = \mathrm{Id} - \mathcal{P}(x^{\perp})$ for any Lagrangian subspace x, we have

$$(\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))^{-1} = (\mathcal{P}(\theta^{\perp}) - \mathcal{P}(\mu^{\perp}))(\mathcal{P}(\tilde{\theta}^{\perp}) - \mathcal{P}(\mu^{\perp}))^{-1}.$$

This gives the first equality (5.9).

Next we prove the coincidence of the first term (5.8) and the third term (5.10), then we know all the term coincide.

From the equality:

$$\begin{aligned}&(\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))^{-1} \cdot (\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) + \mathcal{P}(\mu))^{-1}\\ &\quad = (\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\theta) + \mathcal{P}(\mu))^{-1} \cdot (\mathcal{P}(\theta) + \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) + \mathcal{P}(\mu))^{-1}\end{aligned}$$

we have

$$\begin{aligned}&(\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))^{-1}\\ &\quad = (\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\theta) + \mathcal{P}(\mu))^{-1} \cdot (\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta})\\ &\qquad + \mathcal{P}(\mu))^{-1} \cdot (\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) + \mathcal{P}(\mu))^{-1} \cdot (\mathcal{P}(\theta) + \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) + \mathcal{P}(\mu))^{-1}\\ &\qquad \cdot (\mathcal{P}(\tilde{\theta}) + \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))^{-1}.\end{aligned}$$

When we express the Lagrangian subspace $\theta \subset \mu + J(\mu) = H$ as the graph of an operator $T_{\theta} : \mu \to J(\mu)$, the operator $(\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\theta) + \mathcal{P}(\mu))^{-1}$ is expressed in the following form:

$$(\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\theta) + \mathcal{P}(\mu))^{-1} : \begin{pmatrix} x \\ J(y) \end{pmatrix} \mapsto \begin{pmatrix} -\mathrm{Id} & 0 \\ -T_{\theta} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ J(y) \end{pmatrix}, \quad x, y \in \mu.$$

Hence we see that the operator:

$$(\mathcal{P}(\theta) - \mathcal{P}(\mu))(\mathcal{P}(\theta) + \mathcal{P}(\mu))^{-1} \cdot (\mathcal{P}(\tilde{\theta}) - \mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta}) + \mathcal{P}(\mu))^{-1}$$

is of the form:

$$\begin{pmatrix} \mathrm{Id} & 0 \\ T_{\theta} - T_{\tilde{\theta}} & \mathrm{Id} \end{pmatrix}.$$

When θ and $\tilde{\theta}$ almost coincide, then this operator is of the form "Id + finite rank operator", since $T_{\theta} - T_{\tilde{\theta}}$ is a finite rank operator. Moreover we have

$$\det{}_F \begin{pmatrix} \mathrm{Id} & 0 \\ T_{\theta} - T_{\tilde{\theta}} & \mathrm{Id} \end{pmatrix} = 1.$$

Finally, together with an invariance of the Fredholm determinant with respect to conjugations we have

$$\begin{aligned}&\det_F\{(\mathcal{P}(\tilde{\theta})-\mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta})+\mathcal{P}(\mu))^{-1}\\&\quad\cdot(\mathcal{P}(\theta)+\mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta})+\mathcal{P}(\mu))^{-1}\cdot(\mathcal{P}(\tilde{\theta})+\mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta})-\mathcal{P}(\mu))^{-1}\}\\&\quad=\det_F\{(\mathcal{P}(\theta)+\mathcal{P}(\mu))(\mathcal{P}(\tilde{\theta})+\mathcal{P}(\mu))^{-1}\},\end{aligned}$$

which proves the desired result. □

Remark 5.5. Although we know the triviality of the line bundle $q_\lambda^*(\mathcal{L}_Q)$, there are no natural global trivializations. The *Maslov line bundle* on $\mathcal{F}\Lambda_\lambda$ (we do not define this here, but is defined in a similar way as for the finite dimensional case, see [6]) is also a trivial line bundle just by its definition for which the transition functions are given by the infinite dimensional analog of the *Hörmander indexes* [4]. So it is interesting to give an isomorphism of these two line bundles on a particular subspace in the Fredholm Lagrangian Grassmannian in terms of a certain geometric and/or analytic data, which will give us a relation of the Fredholm determinant and the Maslov index.

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